CBS inequality (Cauchy – Bunyakovskii – Schwarz inequality)

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Question 1

(a) By considering the inequality $(a_i x - b_i)^2 \ge 0$ prove the Cauchy-Schwarz Inequality:

"Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be two sets of real numbers.

$$\left[\sum_{i=1}^{n} a_i b_i\right]^2 \le \left[\sum_{i=1}^{n} a_i^2\right]\left[\sum_{i=1}^{n} b_i^2\right]$$

Equality sign holds iff $a_1 : a_2 : \ldots : a_n = b_1 : b_2 : \ldots : b_n$. "

(b) Given that $a, b, c > 0$ and $a + b + c = 3$. Prove that:

(i) $a^2 + b^2 + c^2 \ge 3$ **(ii)** $a^3 + b^3 + c^3 \ge 3$ **(iii)** $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le 3$

Solution

(a) $(a_i x - b_i)^2 \ge 0$ $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} (a_i x - b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)x^2 - 2\sum_{i=1}^{n} a_i b_i + \left(\sum_{i=1}^{n} b_i^2\right) \ge 0 \qquad \forall x \in \mathbf{R}$$

$\therefore \Delta \le 0$

$$\left(2\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \le 0$$

$$\therefore \quad \left[\sum_{i=1}^{n} a_i b_i\right]^2 \le \left[\sum_{i=1}^{n} a_i^2\right]\left[\sum_{i=1}^{n} b_i^2\right]$$

Equality sign holds $\Leftrightarrow (a_i x - b_i)^2 = 0$ $i = 1, 2, \ldots, n$

$\Leftrightarrow a_i x - b_i = 0$ $i = 1, 2, \ldots, n$

$\Leftrightarrow$ $a_1 : a_2 : \ldots : a_n = b_1 : b_2 : \ldots : b_n$.

(b) **(i)** Apply Cauchy-Schwarz inequality:

$$(a\times 1 + b\times 1 + c\times 1)^2 \le (a^2 + b^2 + c^2)(1^2 + 1^2 + 1^2)$$

$$\therefore a^2 + b^2 + c^2 \ge \frac{1}{3}(a+b+c)^2 = \frac{1}{3}\times 3^2 = 3$$

(ii) $$\left(a^{\frac{3}{2}}a^{\frac{1}{2}} + b^{\frac{3}{2}}b^{\frac{1}{2}} + c^{\frac{3}{2}}c^{\frac{1}{2}}\right)^2 \le \left(a^3 + b^3 + c^3\right)(a+b+c)$$

$$\left(a^2 + b^2 + c^2\right)^2 \le 3\left(a^3 + b^3 + c^3\right)$$

$$\therefore \quad a^3 + b^3 + c^3 \ge \frac{1}{3}\left(a^2 + b^2 + c^2\right)^2 \ge \frac{1}{3}\times 3^2 = 3, \text{ by (b)(i).}$$

(iii) $$\left(\sqrt{a}\sqrt{b} + \sqrt{b}\sqrt{c} + \sqrt{c}\sqrt{a}\right)^2 \le (a+b+c)(b+c+a) \le 3\times 3 = 9$$

$$\therefore \quad \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le 3$$

Question 2

(a) Let $a_1, a_2, \cdots, a_n$ be positive real numbers. By considering the Cauchy-Schwarz inequality, prove that $\sum_{i=1}^{n} a_i \le \sqrt{n\left(\sum_{i=1}^{n} (a_i)^2\right)}$.

(b) Hence, or otherwise, prove that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \le \sqrt{2n-1} \quad \text{for all natural numbers n.}$$

Solution

(a) Applying the Cauchy-Schwarz inequality to the numbers:

$a_1, a_2, ..., a_n$ **;** 1, 1, …, 1

$$\left[\sum_{i=1}^{n} a_i \times 1\right]^2 \le \sum_{i=1}^{n} a_i{}^2 \sum_{i=1}^{n} 1^2$$

$$\therefore \quad \sum_{i=1}^{n} a_i \le \sqrt{n\left(\sum_{i=1}^{n} (a_i)^2\right)}$$

(b) Let $a_i = \frac{1}{i}$, $1 \le i \le n$.

Then by (a), we have

$$\sum_{i=1}^{n} \frac{1}{i} \le \sqrt{n\sum_{i=1}^{n} \frac{1}{i^2}} \le \sqrt{n\left(1+\sum_{i=2}^{n} \frac{1}{i(i-1)}\right)} = \sqrt{n\left(1+\sum_{i=2}^{n} \left(\frac{1}{i-1}-\frac{1}{i}\right)\right)} = \sqrt{n\left(1+1-\frac{1}{n}\right)}$$

$$\therefore \quad 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \le \sqrt{2n-1}$$